Universality class for a one-dimensional evolution model

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(Received 19 December 1996)

We present numerical evidence that avalanche dynamics in the evolution model has the same universality class as the diffusion equation $\partial_t p = x^{-\alpha}\partial_{xx}p + vx^{-\alpha-1}\partial_x p$. Numerically we measure the exponent α and the drift v and, using the relations provided by the theory of the diffusion equation, we compute the avalanche critical exponent τ and the mass dimension exponent D. The computed values agree with the previous numerical results. [S1063-651X(97)05609-2]

PACS number(s): 05.40.+j, 87.10.+e, 64.60.Lx

Bak and Sneppen (BS) [1] formulated a coarse-grained description of biological evolution. It originates from models which tried to find an adaptive dynamics of the genotype in the space of mutations [2]. Using a simple local dynamical rule the model is able to reach a stationary state in which the distribution of large size events is scale invariant. From this property one can argue that extinction events involving a large number of species are characteristic of the internal rules of the system and not produced by external agents. The mathematical simplicity of the model attracted numerical studies [1,3,4] and mean-field analytic treatments [5–7].

The model treats a number of N species interacting on a one-dimensional chain viewed as a simple picture of the food chain. Each species has an assigned scalar parameter or fitness with values in the (0,1) interval, thought to be a measure for the adaptability of species to the ecosystem. One time step of the dynamics consists of choosing the site with the smallest fitness, then new random independent values from the (0,1) interval are attributed to this site and to its two neighbors with a uniform distribution. We define a λ avalanche ($0<\lambda<1$) of size t, the event having t time steps between two consecutive configurations with all the fitness values greater than λ. For N large the one-site stationary distribution is vanishingly small for $\lambda\approx 0.667$, and constant above this value [1]. The avalanche distribution for $\lambda\approx 0.667$ decays algebraically for large t; the spatial correlation of the activity and the spatial structure of the avalanches also have scale invariant properties [3,4,8]. Numerical simulation and the scaling the ansatz suggest that the model is described by two independent critical exponents; the scaling relations yield the others [8]. The following two exponents are of interest for us: the critical exponent τ of the temporal distribution of the avalanches

$$p(t)\approx t^{-\tau}, \quad t\to\infty, \tag{1}$$

and the avalanche mass dimension critical exponent D which describes the temporal evolution of the average avalanche spatial width

$$\overline{w(t)}\approx t^{1/D}. \tag{2}$$

The spatial width of an avalanche is defined as the number of sites between the leftmost site with fitness less than λ, and the rightmost site with fitness less than λ. An avalanche ends when its spatial size is zero. The site with the smallest fitness is also called active.

In this paper we present numerical evidence that the avalanche width dynamics is described asymptotically by a Markovian continuous time random walk (CTRW). If the active site is at one of the extrema of the avalanche or next to this site, the probability that the avalanche increases or decreases its width by one or two steps sites is determined by the dynamic rule of the system, and has no memory effect by the very rule of the model; on the other hand, the probability that the avalanche reduces its width by more than two sites depends upon the probability that next to the extreme active site there is an interval of a given width with all the fitness higher than λ. The distribution of the ''empty'' intervals will self-average in time; therefore for large times there is a stationary probability to jump backward from a given width to another one. These arguments make plausible the idea that a Markovian CTRW can describe the asymptotic behavior of the avalanche width in the BS model. Since we are interested in asymptotic behavior, we take the continuum limit. In this frame a Markovian CTRW is described by a diffusion equation [9] of the type

$$\frac{\partial p}{\partial t} = a(x)\frac{\partial^2 p}{\partial x^2} + b(x)\frac{\partial p}{\partial x}, \tag{3}$$

where $a(x)$ and $b(x)$ are the local variance and the local drift obtained through the limit procedure from the CTRW.

In the BS model the variance $a(x)$ decreases as the width of an avalanche increases, since the activity takes longer and longer time intervals to reach one of the extrema, (Fig. 3); in this condition avalanches of all sizes appear if the drift vanishes as the width increases [10]. This can also be understood intuitively, recalling the fact that for a simple diffusion process, $a(x)=\text{const}$, a constant symmetric drift $b(x)=-|v|$ bonds the diffusing particle close to the origin [9]; if the local variance is a decreasing function of x, one can also

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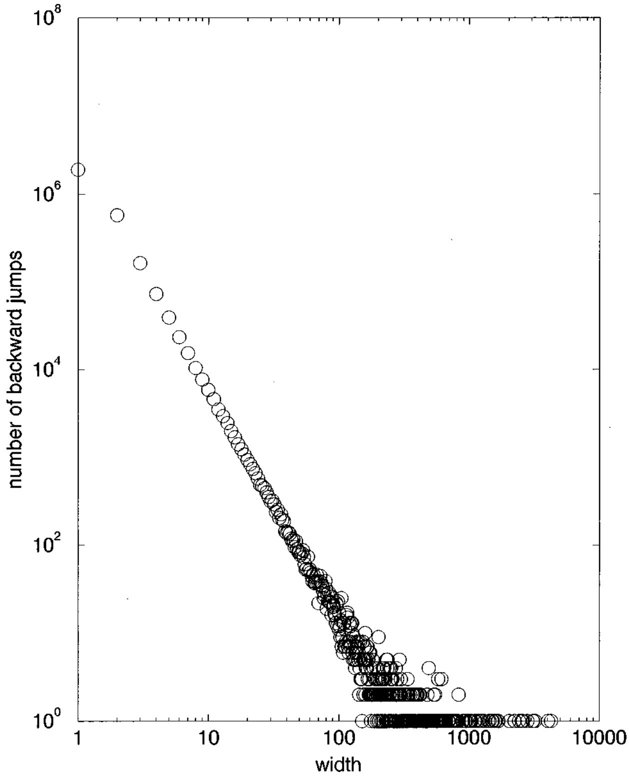

FIG. 1. The distribution of the backwards jumps of the avalanche width. The statistics were collected from avalanches with the spatial size higher than 4000 sites. The system size was 8192.

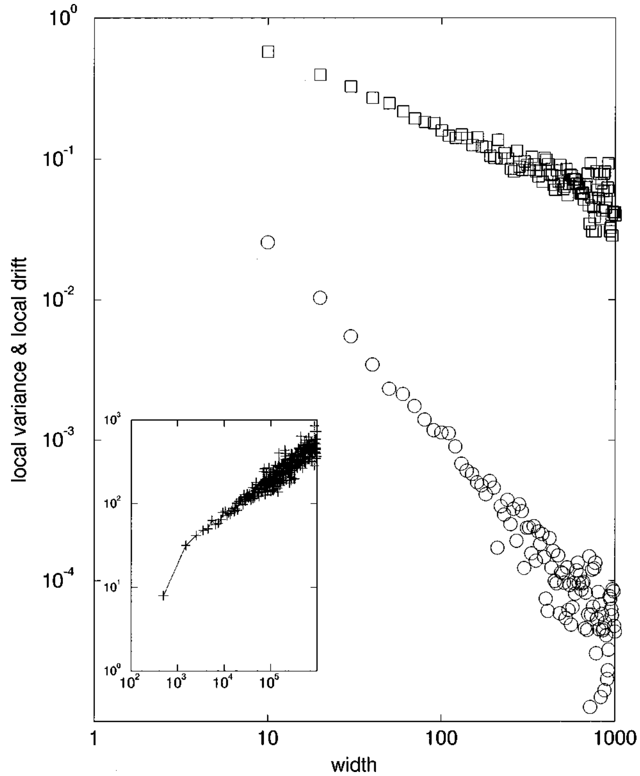

FIG. 2. The local drift and the local variance against avalanche width. In the inset the average maximal width against avalanche temporal size.

infer that the local drift has to decrease fast enough to allow large-scale events. We impose the null drift condition, seeking an asymtotically zero value for the mean displacement,

$$\lambda = \lim_{n\to\infty} \int^{n} p_n(i) i, \tag{4}$$

where $p_n(i)$ is the transition probability of a backward jump if the avalanche width is n. The null drift condition yields a lower bound for $\lambda_{\text{critical}}$. Considering only the first two backward steps whose probability transition can be computed, we have

$$\lambda = 3(1-\lambda)^2\lambda - (1-\lambda)\lambda^2,$$

with the positive solution $\lambda_{\text{critical}} = 0.5$. A backward jumps higher than two steps will increase $\lambda_{\text{critical}}$.

If $\lambda = \lambda_{\text{critical}}$ the distribution of the backward jump has an algebraic tail, as Fig. 1 shows; hence for a finite avalanche width i we have the drift proportional to

$$\int_{n}^{\infty} p(i) i \approx n^{-\beta+2}, \tag{5}$$

where $p(i) = \lim_{n\to\infty} p_n(i)$. This scale invariant structure will imply that the local variance of the transition probabilities will behave as

$$\lambda + \int^{n} p(i) i^2 \approx c + d n^{-\beta+3}, \quad n \gg 1, \tag{6}$$

with c, d, and β constants. The asymptotic behavior of the coefficients of Eq. (3) is obtained by dividing Eqs. (5) and (6) by the mean lifetime of an avalanche at each site. Numerically, (Fig. 3), we found that the mean lifetime grows linearly,

$$\overline{t_n} \approx n^p, \quad p = 1.00 \pm 0.01.$$

Hence the general equation describing the asymptotic behavior of the BS model is

$$\frac{\partial p}{\partial t} = \frac{K'}{x^p} \frac{\partial^2 p}{\partial x^2} + \frac{K}{x^\alpha} \frac{\partial^2 p}{\partial x^2} + \frac{v}{x^{\alpha+1}} \frac{\partial p}{\partial x} \tag{7}$$

where K, K', and v are positive constants and $\alpha = \beta - 3 + p$. A change of scale $t \to ct$, $x \to c^{1/(\alpha+2)} x$ transforms the above equation to

$$\frac{\partial p}{\partial t} = c^{1-(p+2)/(p+\beta-1)} \frac{K'}{x^p} \frac{\partial^2 p}{\partial x^2} + \frac{K}{x^\alpha} \frac{\partial^2 p}{\partial x^2} + \frac{v}{x^{\alpha+1}} \frac{\partial p}{\partial x}.$$

This shows that the operator

$$\frac{K'}{x^p} \frac{\partial^2}{\partial x^2}$$

is vanishing as time increases ($c \to \infty$) if $\beta < 3$. Consequently we can use the Green function of the unperturbed operator to derive the critical exponents of the asymptotic behavior. The

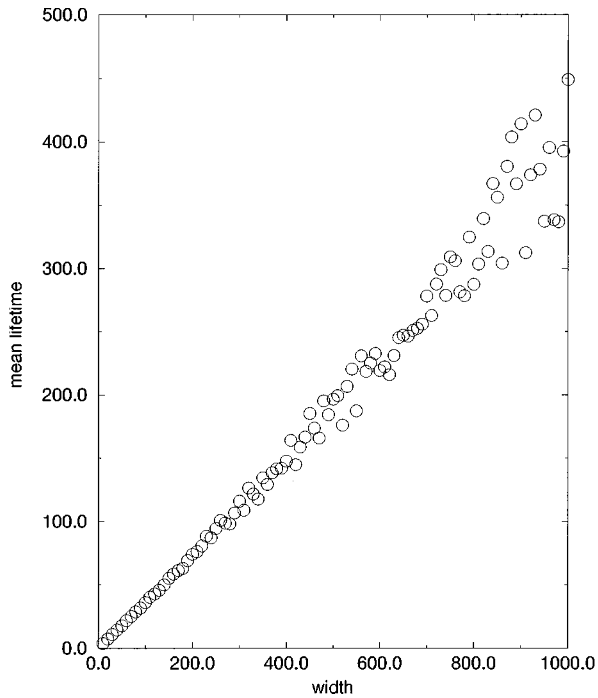

FIG. 3. The characteristic lifetime (in arbitrary units) of an avalanche at a given width. This quantity is proportional with the average number of steps in which the activity touches the extreme sites of an avalanche.

critical exponent of the avalanche temporal distribution τ can be obtained from the first return time distribution. In Ref. [10] we showed that the first return in the origin critical exponent has the expression

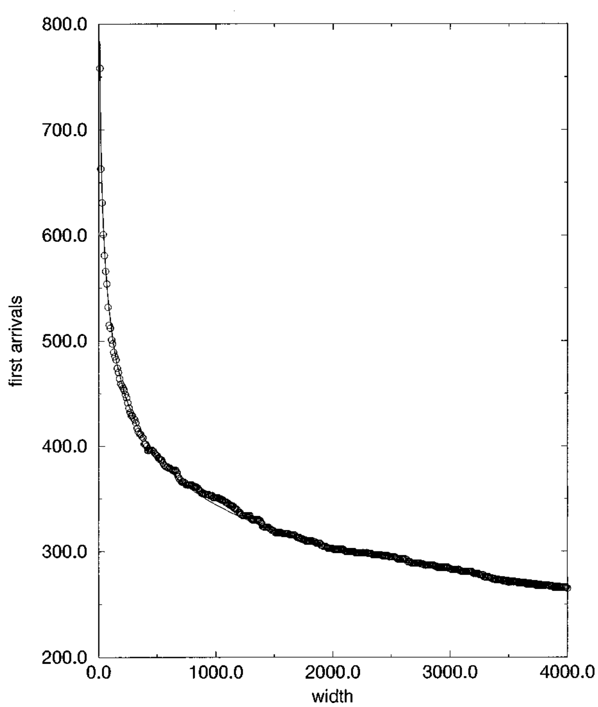

FIG. 4. The number of first arrivals to a width $y>x$ without passing through the state with zero width. This quantity is predicted to behave algebraically by Eq. (10). The fit with a power function (the continuous line) is very good.

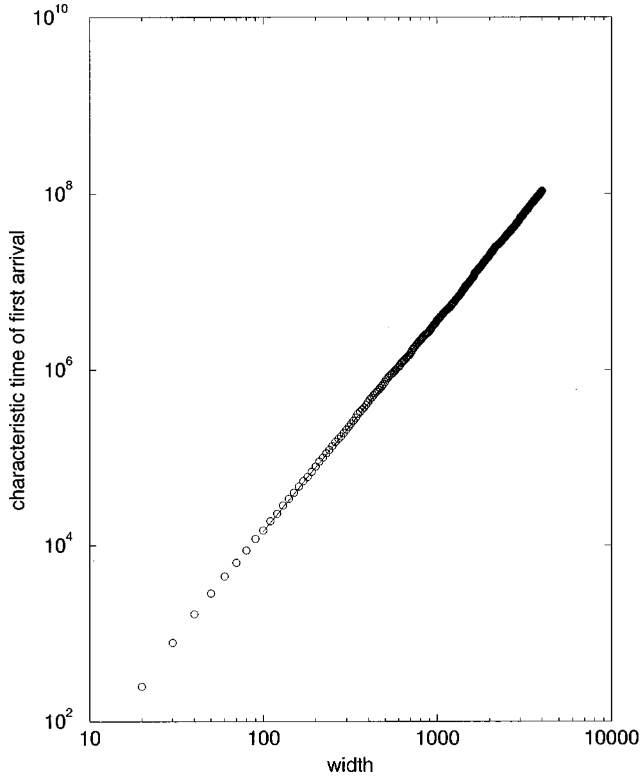

FIG. 5. The characteristic time (in arbitrary units) in which an avalanche touches the first time width y starting from a width $x<y$ without passing through the state of zero width. Equation (10) predicts $\overline{t_{xy}}\approx y^{\alpha+2}-x^{\alpha+2}\approx y^{\alpha+2}$ for $y\gg x$. The graph confirms the algebraic behavior.

$$\tau=1+\frac{1-v/K}{\alpha+2}. \tag{8}$$

The avalanche mass dimension exponent D can be obtained from the asymptotic behavior in time of the average width [10]:

$$w_{\text{average}}=\frac{\int_0^\infty yG_t(x,y)dy}{\int_0^\infty G_t(x,y)dy}\sim t^{1/(\alpha+2)}. \tag{9}$$

We numerically obtained the local variance and the local drift for the one-dimensional Bak-Sneppen model in the interval (10,100). This corresponds to an avalanche maximal temporal size of order 10^6 as the average maximum width scales $\overline{w_{\max}}\approx t^{0.42}$ (Fig. 2). The average lifetime of an avalanche at a given width increases linearly, as Fig. 3 shows. The critical exponent β, defined in Eq. (5), was measured from the local drift data and from the asymptotic behavior of the backward jumps distribution; see Figs. 2 and 1. Their value agrees with $\beta=2.40\pm0.05$. Using Eq. (9), for the mass dimension exponent $D^{-1}=0.41\pm0.02$ we obtain a value which is in agreement with the previous measurements [3,8]. The precise measurement of the constants v and K is difficult due to the high weight of the short backwards jumps in respect to the long ones, (Fig. 1). Hence the number of steps needed to obtain the distribution at every avalanche width is very large.

There is an indirect but precise method of measuring the exponents α and $1-v/K$ using the Laplace transform of the probability of first arrival in y stating from $x<y$ without touching the origin. For small λ we have [10]

$$p_\lambda^{\text{first}}(x,y) \sim \left(\frac{x}{y}\right)^{1-(v/K)} \left(1 - \frac{1}{(\alpha+2)^2(\nu+1)} \times (y^{\alpha+2} - x^{\alpha+2})\lambda\right), \tag{10}$$

where $\alpha=3-\beta+p$, $y>x$. One can see that if $\lambda=0$, the previous formula gives the probability for an avalanche to reach, for the first time, a width y starting from the width x without passing through the state with zero width. Numerical simulation gives an algebraic behavior (see Fig. 4), and we obtained $1-v/K=0.180\pm0.006$. The constant multiplying by λ in Eq. (10) is the characteristic time for an avalanche to touch width y for the first time starting from width x. Numerically we found that it behaves as an algebraic function of width and $\alpha+2=2.42\pm0.02$ (Fig. 5). This two exponents allow us to compute the critical exponents of the avalanche size distribution using Eqs. (8) and (9). We obtain $\tau=1.074\pm0.004$ and $D^{-1}=0.413\pm0.008$. These values are in good agreement with the previous determined ones in the Ref. [4], $\tau=1.074\pm0.004$, and $D^{-1}=0.4114\pm0.0020$.

From the above numeric analysis we conclude that the diffusion equation (7) describes the universality class for the BS models. The basic facts leading to this description are as follows.

(1) The underlying Markov chain reaches the stationary state at any finite width of the avalanche; therefore one can think in terms of a stationary probability transition from one width to another one.

(2) The probability that the activity remains inside of an avalanche decreases exponentially in time. This is the second ingredient needed to complete the Markovian CTRW picture.

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